K. Ueda – 29/5/08 – Toric degenerations of Gelfand-Cetlin systems & potential f^{ns}

(w/ Nishinou & Nohara)

<u>Gelfand-Cetlin system</u> = integrable system on flag manifold (seen as a coadjoint orbit)

$G = U(n)$

$\mathfrak{g} = \mathrm{Lie}(G) \xrightarrow[\text{metric}]{\sim} \mathfrak{g}^*$

$\lambda = \mathrm{diag}(\lambda_1 \dots \lambda_n), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

$G_\lambda = Ad_G \lambda$ coadjoint orbit

is a <u>complex manifold</u> : $G_\lambda \cong P_\lambda \backslash GL_n(\mathbb{C})$ ← parabolic subgp

(partial flag variety)

- for $\lambda_1 > \dots > \lambda_n$, $G_\lambda \cong B \backslash GL_n(\mathbb{C})$ full flag manifold

- G_λ is a <u>symplectic manifold</u> : $\omega_\lambda(ad_\xi(x), ad_\eta(x)) = \langle x, [\xi, \eta]\rangle$

Kostant-Kirillov form

- $\lambda_i \in \mathbb{Z}\ \forall i \Leftrightarrow \exists L \to G_\lambda$ Hermitian holom. line bundle st. $c_1(L) = \omega_\lambda$

- For $x \in G_\lambda$, let $x^{(i)}$ be its $(i \times i)$ lower-right submatrix $\left(\begin{array}{c|c} & \\ \hline & x^{(i)} \end{array}\right)\}i$

x Hermitian $\Rightarrow x^{(i)}$ Hermitian

$\Rightarrow$ has real eigenvalues $\lambda_1^{(i)} \geq \dots \geq \lambda_n^{(i)}$

<u>Thm</u> (Guillemin-Sternberg)

The image of $\Phi : G_\lambda \longrightarrow \mathbb{R}^{n(n-1)/2}$ is the <u>Gelfand-Cetlin polytope</u>

$x \longmapsto \left(\lambda_j^{(i)}\right)_{\substack{i=1,\dots,n-1 \\ j=1,\dots,i}}$ $\qquad \Delta_\lambda := \Phi(G_\lambda)$

$$\Delta_\lambda = \left\{ \begin{array}{ccccccccc} \lambda_1 \geq \lambda_1^{(n-1)} \geq \lambda_2 \geq \lambda_2^{(n-1)} \geq \lambda_3 \geq & \dots & \lambda_{n-1} \geq \lambda_{n-1}^{(n-1)} \geq \lambda_n \\ \geq \lambda_1^{(n-2)} \geq \quad \geq & \dots & \geq \\ \ddots & & \\ \geq \lambda_1^{(1)} \geq & \dots & \end{array} \right\}$$

★ <u>Geometric quantization</u> $\Rightarrow H^0(G_\lambda, L_\lambda) \cong \mathrm{span}_{\mathbb{C}}\left(\Delta_\lambda \cap \mathbb{Z}^{n(n-1)/2}\right)$

- <u>E.g.</u>: $(\lambda_1 \dots \lambda_4) = (1,1,0,0)$ $\quad (\rightsquigarrow G_\lambda = Gr(2,4))$

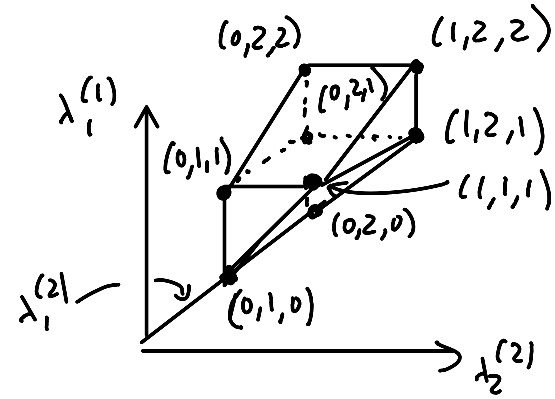
$$\begin{array}{ccccccc} 1 & & 1 & & 0 & & 0 \\ & \geq 1 \geq & & \geq \lambda_2^{(3)} \geq & & \geq 0 \geq & \\ & & \geq \lambda_1^{(2)} \geq & & \geq \lambda_2^{(2)} \geq & & \\ & & & \geq \lambda_1^{(1)} \geq & & & \end{array}$$

Necess. $\lambda_1^{(1)} = 1$, $\lambda_3^{(1)} = 0$

so $\dim \Delta_\lambda = 4 = \dim_{\mathbb{C}} Gr(2,4)$. ✓

- <u>Eg</u>: for $(\lambda_1, \lambda_2, \lambda_3) = (2,1,0)$

$$\begin{array}{ccccc} 2 & & 1 & & 0 \\ & \geq \lambda_1^{(2)} \geq & & \geq \lambda_2^{(2)} \geq & \\ & & \geq \lambda_1^{(1)} \geq & & \end{array}$$



In this example, $u \in \mathring{\Delta}_\lambda \Rightarrow \phi^{-1}(u) \simeq T^3$

but $\phi^{-1}(1,1,1) = \left\{ \begin{pmatrix} 1 & z & w \\ \bar{z} & 1 & 0 \\ \bar{w} & 0 & 1 \end{pmatrix} \middle| \, |z|^2 + |w|^2 = 1 \right\} \simeq S^3$

<u>Thm</u>: (Gonciulea–Lakshmibai, Kogan–Miller, ...)

$\exists\, f: \mathfrak{X} \to \mathbb{C}$ flat family s.t. $f^{-1}(t) \cong G_\lambda$ for $t \neq 0$

and $f^{-1}(0)$ is the toric var. associated with Δ_λ

This is given by a deformation of Plucker relations:

e.g. let $\mathfrak{X} = \left\{ ([x:y:z], [u:v:w], t) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{C} \,/\, tux + yv + zw = 0 \right\}$

then $\mathfrak{X}_t \cong G_{(2,1,0)}$ for $t \neq 0$

$\mathfrak{X}_0 = \{yz + zw = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \simeq$ cone over $\mathbb{P}^1 \times \mathbb{P}^1$ (ODP singularity)

* in fact the S^3 we saw in $\phi^{-1}(1,1,1)$ is the vanishing cycle of the ODP degeneration !!!

<u>Main goal:</u> Compute the potential function (in sense of FOOO) for $L(u) := \phi^{-1}(u)$, $u \in \mathring{\Delta}_\lambda$.

<u>Recall:</u>

$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \,/\, a_i \in \mathbb{Q},\ \lambda_i \geq 0,\ \lim_{i\to\infty} \lambda_i = +\infty \right\}$ Novikov ring

$\Lambda_+ \subset \Lambda_0$ maximal ideal $(= \{1 + \dots\})$

$\{m_k\}_{k=0}^{\infty}$ canonical model of Fukaya's A_∞-structure on $H^*(L(u); \Lambda_0)$

$b \in H^1(L(u), \Lambda_+)$ is said to be a <u>weak bounding cochain</u> if

$$\exists\ P(b) \in \Lambda_0 \ \text{ s.t. } \ \sum_{k=0}^{\infty} m_k(b, \dots, b) = P(b) \cdot PD([L(u)])$$

(ie. $(L(u), m_k^b)$ is weakly unobstructed)

$P =$ potential function: $\{\text{weak } \partial \text{ cochains}\} \subset H^1(L(u), \Lambda_+) \to \Lambda_0$

<u>Thm:</u> Any element $x \in H^1(L(u), \Lambda_+)$ is a weak bounding cochain and $P(x) = \sum_{i=1}^{m} e^{\langle v_i, x \rangle} T^{l_i(u)}$

where $m = \#$ facets of Δ_λ ; $v_i =$ primitive normal vectors;

$\Delta_\lambda = \{ u \in \mathbb{R}^n \,/\, l_i(u) = \langle v_i, u \rangle - \tau_i \geq 0,\ i = 1..m \}$

<u>Note:</u> G_λ is Fano ; calculation proceeds via toric degeneration and using Cho-Oh (<u>note</u>: our toric var. may be singular, but that doesn't affect counting of Maslov 2 discs). The thm. says things work as in the toric case ...

Put $y_i = e^{x_i} T^{u_i}$ and $q_i = T^{\lambda_i}$, then

$$P(x) \in \mathbb{Q}[\underbrace{q_1^{\pm 1}, \dots, q_N^{\pm 1}}_{\text{Kähler params.}}][\underbrace{y_1^{\pm 1} \cdots y_N^{\pm 1}}_{\text{coords. on mirror}}]$$

<u>Ex:</u> $\lambda = (\lambda_1 > \lambda_2 > \lambda_3)$

$$\Rightarrow P = \frac{q_1}{y_1} + \frac{y_1}{q_2} + \frac{q_2}{y_2} + \frac{y_2}{q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2}$$

Jacobian ring has rank $J(P) = 6 = \dim H^*(G_\lambda)$
(6 distinct crit pts = balanced T^3's inside Δ_λ)

however vol(Newton poly.) = 8

so a generic perturb^{n} acquires 2 extra crit pts ???

<u>NB:</u> the exc. $S^3 \to (1,1,1)$ is <u>not</u> balanced ($HF = 0$)
(believe: it bounds a Maslov 4 disc through 2 generic pts
$\Rightarrow \delta^{(4)}(pt) = [S^3]$ kills it)

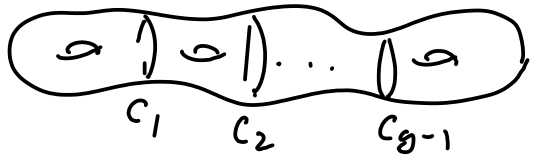
<u>Ex:</u> $\lambda = (\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4)$

$$\rightsquigarrow P = \frac{q_1}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_3}{q_3} + \frac{y_2}{y_4} + \frac{y_4}{y_3}$$

rank $QH^*(G_\lambda) = 6 \neq 4 = \text{rank } Jac(P)$

$\uparrow$ guess: difference means two of the Lagrangian fibers on $\partial\Delta_\lambda$ ($\simeq S^1 \times S^2$) are balanced.

- N_g = moduli space of flat $SU(n)$-bundles on a genus g curve

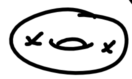


$T = T^{(n-1)(g-1)} \curvearrowright N_g$ Goldman's twist action

(twist by elt of $T^{n-1}_{diag} \subset SU(n)$ along c_i's)

$\mu : N_g \longrightarrow \mathrm{Lie}\, T$ moment map (= holonomies on c_i)

$\mu^{-1}(\alpha)/T \cong N(\alpha)$ = moduli space of parabolic bundles on

where $N\Big($  $\Big) \simeq G_{\lambda_1} \times G_{\lambda_2}$ ... so ...

<u>Problem:</u> Can one obtain an integrable system on N_g, admitting toric degenerations by combining Goldman functions with Gelfand-Cetlin systems?